Period doubling when accelerating alternating Schwarz with Newton-Raphson

Conor McCoid, supervised by Martin J. Gander

University of Geneva

Alternating Schwarz (AltS)

Consider the nonlinear problem $F(u) = 0$ on the domain $[a, b]$ with Dirichlet boundary conditions: $u(a) = A$ and $u(b) = B$. Rather than solve the problem on the entire domain, one may split the domain in two and solve the problem on each subdomain:

$$x \in [a, \beta] \begin{cases} F(u_1^n) = 0 \\ u_1^n(a) = A \\ u_1^n(\beta) = u_2^{n-1}(\beta) \end{cases} \quad (1)$$

$$x \in [\alpha, b] \begin{cases} F(u_2^n) = 0 \\ u_2^n(b) = B \\ u_2^n(\alpha) = u_1^n(\alpha) \end{cases} \quad (2)$$

AltS stops when the value of $u_2^{n-1}(\beta)$ remains constant (up to a given tolerance).

AltS as a fixed point iteration

AltS may be thought of as a fixed point iteration:

$$\begin{cases} F(u_1^n) = 0 \\ u_1^n(a) = A \\ u_1^n(\beta) = \gamma_n \end{cases} \Rightarrow \begin{cases} F(u_2^n) = 0 \\ u_2^n(b) = B \\ u_2^n(\alpha) = u_1^n(\alpha) \end{cases}$$

with $\gamma_{n+1} = u_2^n(\beta)$. The process that transforms γ_n into γ_{n+1} is an implicit function $G(\gamma)$, so that $\gamma_{n+1} = G(\gamma_n)$. $G(\gamma)$ may not have a closed form expression for nonlinear F.

Newton preconditioning (NP)

To speed up convergence, we can apply Newton-Raphson to the function $G(\gamma) - \gamma$. This requires knowledge of $G'(\gamma)$, which we can find by solving the following linear problems:

$$\begin{cases} J(u_1^n) \cdot g_1 = 0 \\ g_1(a) = 0 \\ g_1(\beta) = 1 \end{cases} \quad \begin{cases} J(u_2^n) \cdot g_2 = 0 \\ g_2(b) = 0 \\ g_2(\alpha) = g_1(\alpha) \end{cases} \quad (3)$$

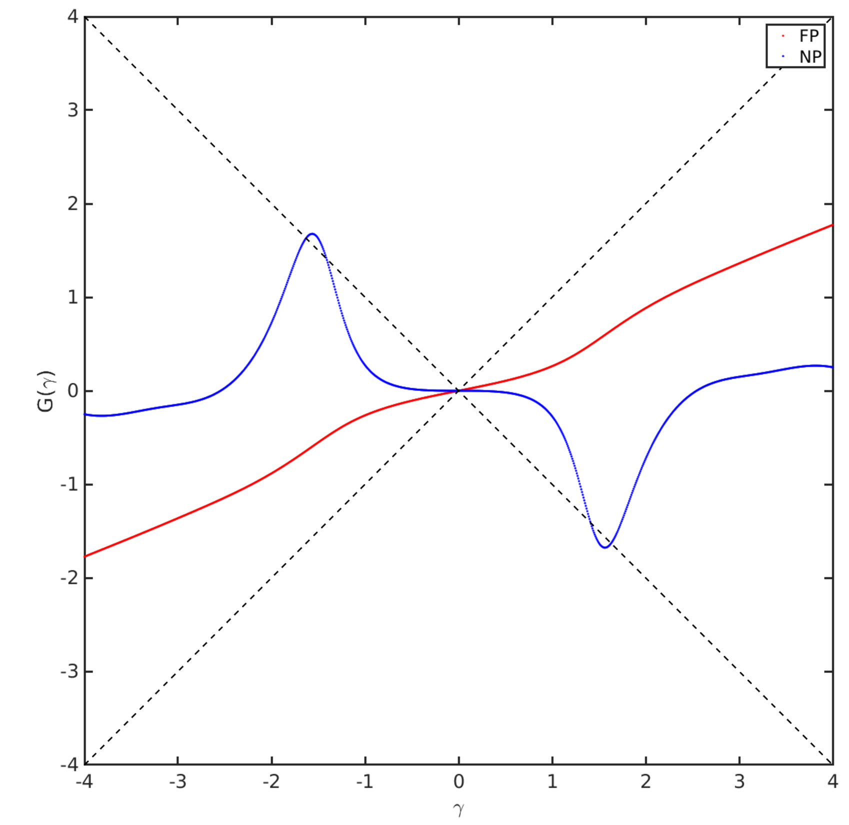
where $J(u)$ is the Jacobian of F evaluated at the function $u(x)$. The derivative of $G(\gamma)$ is then $G'(\gamma) = g_2(\beta)$.

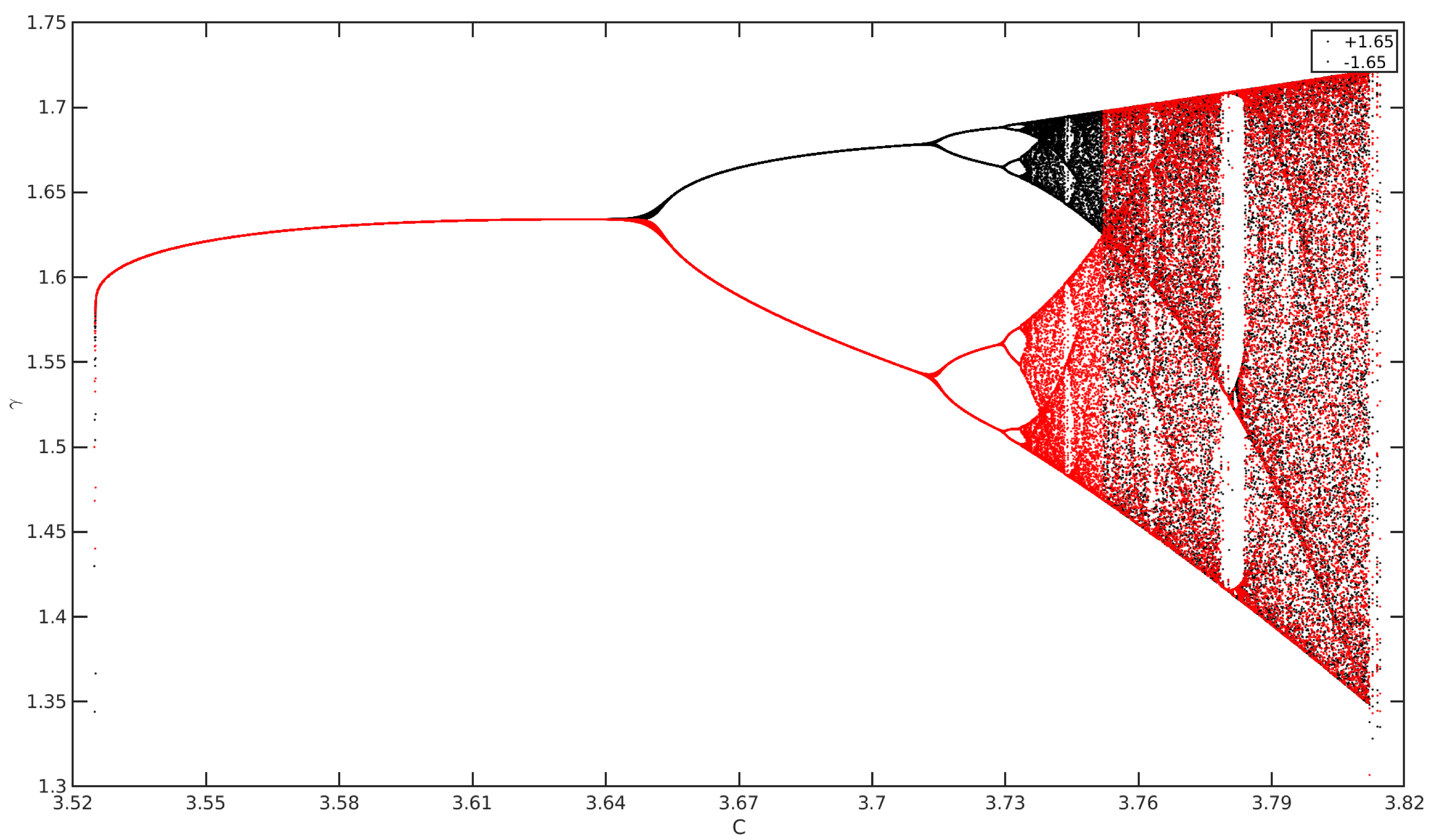
Example

Consider the following example:

$$\begin{cases} u''(x) - \sin(Cu) = 0, \quad x \in [-1, 1], \\ u(-1) = u(1) = 0. \end{cases} \quad (4)$$

The solution is the zero function ($\gamma^* = 0$).
The AltS fixed point function $G(\gamma)$ for this example cannot be written explicitly. It is plotted in the figure to the right (red) alongside its NP counterpart (blue). When the NP function crosses the line $y = 2\gamma^* - \gamma$ cycling becomes possible.





Parameters

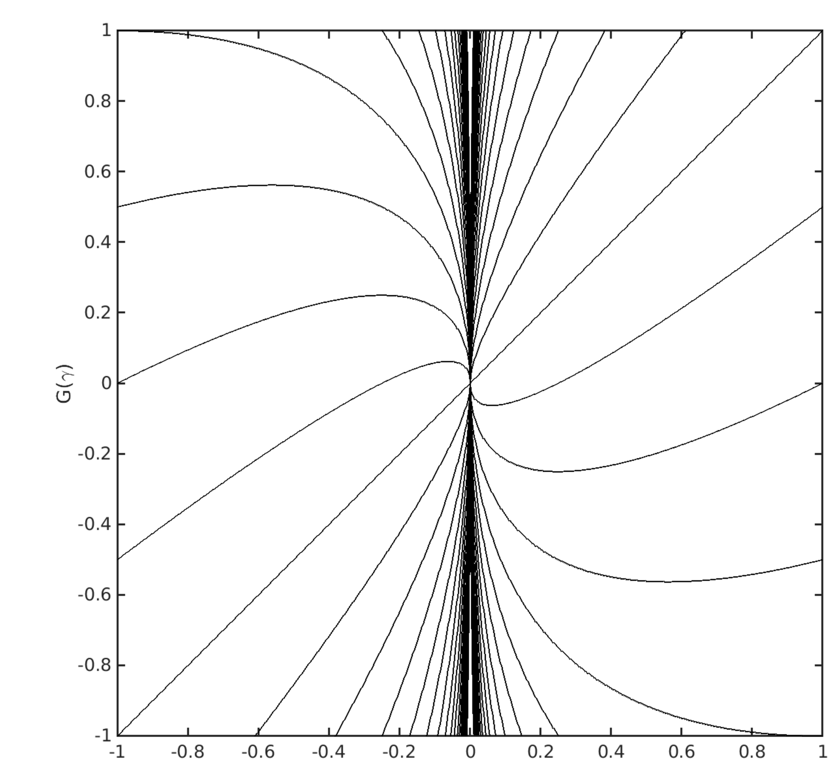
- $\alpha = -0.2$, $\beta = 0.2$;
- $\gamma_0 = \pm 1.65$, $C \in [3.52, 3.82]$;
- 50 iterations of AltS with NP are calculated to achieve stability for each value of C;
- 64 iterations are then plotted.

Results

The results are presented in the figure above. Not included is the graph's reflection over the line $\gamma = 0$ with colours reversed. This behaviour requires specific choices of γ_0. The cycling intervals of γ_0 and C change depending on overlap and transmission conditions in AltS.

Cycling in NP

Cycling under NP cannot occur if the function $G(\gamma)$ is monotonic with respect to the geometry of the figure below. The origin of this figure is the fixed point, (γ^*, γ^*). In particular, if $G''(\gamma) > 0$ for $\gamma < \gamma^*$ and $G''(\gamma) < 0$ for $\gamma > \gamma^*$ then NP will converge quadratically regardless of γ_0.



Cycling becomes much more likely when $G''(\gamma) = 0$ for $\gamma \neq \gamma^*$. The simplest way to ensure this is for the Hessian of $F(u)$ to be zero for $u \neq u^*$, the exact solution.

References

[1] Xiao-Chuan Cai and David E Keyes. Nonlinearly preconditioned inexact Newton algorithms. *SIAM Journal on Scientific Computing*, 24(1):183–200, 2002.

[2] Victorita Dolean, Martin J Gander, Walid Kheriji, Felix Kwok, and Roland Masson. Nonlinear preconditioning: How to use a nonlinear Schwarz method to precondition Newton's method. *SIAM Journal on Scientific Computing*, 38(6):A3357–A3380, 2016.

Acknowledgements

I'd like to thank Martin Gander for his help and direction on this research.