Cycles in Newton-Raphson-accelerated Alternating Schwarz

Conor McCoid, Martin J. Gander

University of Geneva

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Alternating Schwarz (AltS) for a 1D BVP

$$(1)\begin{cases} F(x, u_1, u_1', u_1'') = 0 \\ u_1(a) = A \\ u_1(\beta) = \gamma_n \end{cases} \qquad (2)\begin{cases} F(x, u_2, u_2', u_2'') = 0 \\ u_2(\alpha) = u_1(\alpha) \\ u_2(b) = B \end{cases} \tag{1}$$
$$(3)\gamma_{n+1} = u_2(\beta) = G(\gamma_n)$$

AltS can be thought of as a fixed point iteration, $\gamma_{n+1} = G(\gamma_n)$.

Newton-Raphson Accelerated AltS / Nonlinear Preconditioning

$$(1)\begin{cases} F(x, u_1, u_1', u_1'') = 0 \\ u_1(a) = A \\ u_1(\beta) = \gamma_n \end{cases} \qquad (2)\begin{cases} F(x, u_2, u_2', u_2'') = 0 \\ u_2(\alpha) = u_1(\alpha) \\ u_2(b) = B \end{cases}$$

$$(3)\begin{cases} J(u_1)\cdot(g_1, g_1', g_1'') = 0 \\ g_1(a) = 0 \\ g_1(\beta) = 1 \end{cases} \qquad (4)\begin{cases} J(u_2)\cdot(g_2, g_2', g_2'') = 0 \\ g_2(\alpha) = 1 \\ g_2(b) = 0 \end{cases}$$

$$(5)\gamma_{n+1} = \gamma_n - \frac{u_2(\beta) - \gamma_n}{g_1(\alpha)g_2(\beta) - 1} \quad = \gamma_n - \frac{G(\gamma_n) - \gamma_n}{G'(\gamma_n) - 1} \tag{2}$$

Example

Consider the following second order nonlinear differential equation with homogeneous Dirichlet boundary conditions on the domain (-1,1):

$$u''(x) - \sin(au(x)) = 0.$$

This equation is nonsingular and admits only the trivial solution $u(x) = 0$.

Example

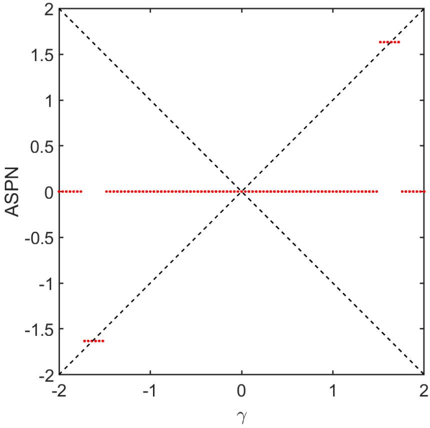

Figure: Newton-Raphson accelerated AltS on 1D sine example, $a = 3.6$ with overlap 0.4. Also plotted is $y = x$ and $y = -x$.

Example

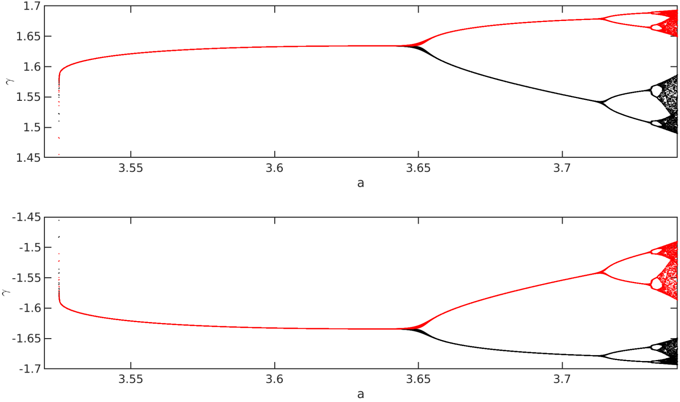

Figure: Period doubling bifurcation in the example caused by NR acceleration.

When does a fixed point iteration (FP) converge in 1D?

Convergence of the iteration $x_{n+1} = g(x_n)$ depends on which region $(x, g(x))$ lies.

1: Monotonic divergence
2: Monotonic convergence
3: Oscillatory convergence
4: Oscillatory divergence

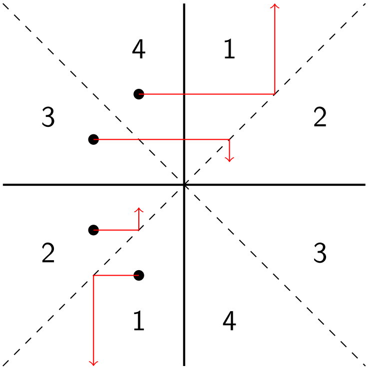

Figure: Behaviour of FP; the origin is the fixed point of the function

When does Newton-Raphson (NR) converge in 1D?

For Newton-Raphson there are no regions. Instead, convergence at a given point is determined by where the slope points. These 'regions' correspond to those for FPI.

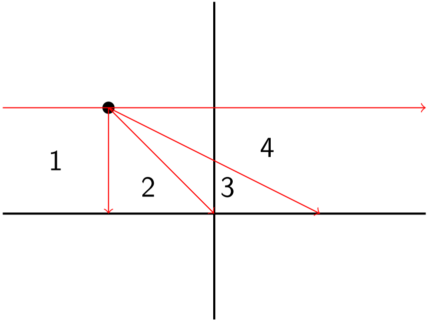

Figure: Regions of NR; the origin is the root of the function

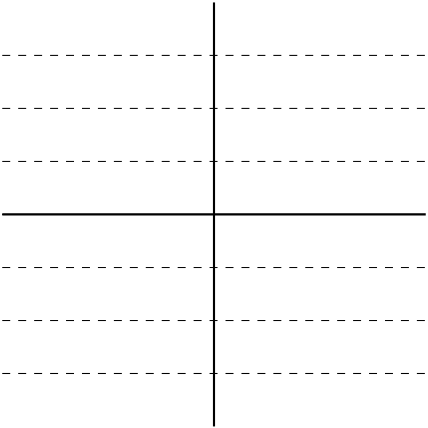

Figure: Tracing border between regions 1 and 4.

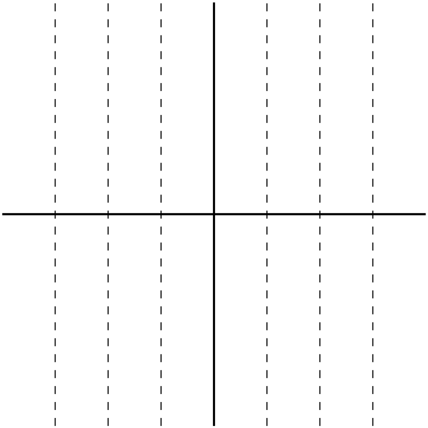

Figure: Tracing border between regions 1 and 2.

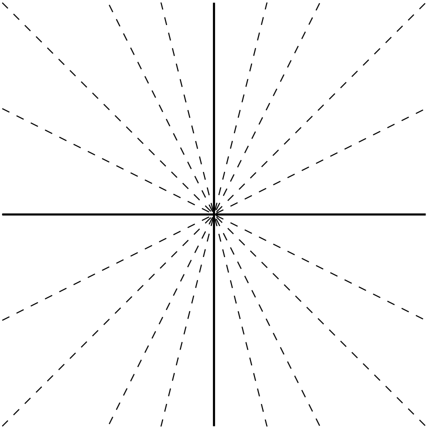

Figure: Tracing border between regions 2 and 3.

What about tracing the border between regions 3 and 4?
Let x^* be a root of $f(x)$ and let $f_C(x)$ trace the border between regions 3 and 4. Then

$$2x^* - x = x - \frac{f_C(x)}{f_C'(x)},$$
$$\implies f_C'(x) = -\frac{f_C(x)}{2(x^* - x)},$$
$$\implies f_C(x) = C\sqrt{|x - x^*|}$$

for any value of $C \in \mathbb{R}$.

When does NR converge in 1D?

We transform from Cartesian coordinates (x, y) to (x, C), where $C = y/\sqrt{|x - x*|}$ (with $x*$ being the root of the function). A function must be monotonic in this geometry for NR to converge.

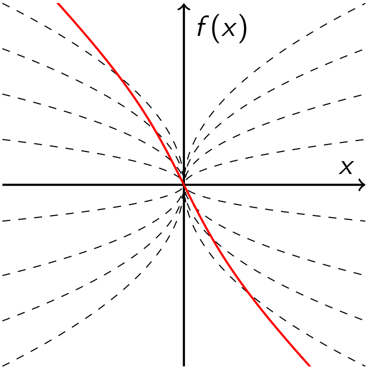

Figure: Geometry of NR; the origin is the root of the function

NR accelerated FP in 1D

Figure 9 is the same as figure 8 tilted so that the x–axis is set to the line $y = x$. A function $g(x)$ must now be monotonic in this geometry if NR on $g(x) - x$ is to converge.

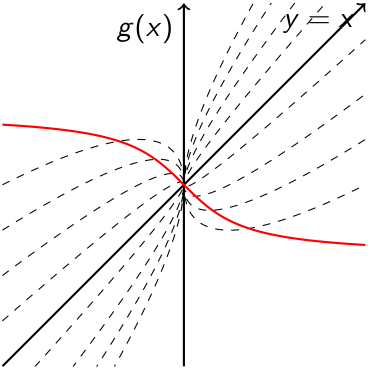

Figure: Geometry of NR of FP; the origin is the root of the function

When does AltS converge in 1D?

In 1D, AltS takes information at a point on the interface and produces an update at that point. This can be thought of as a fixed point function, $G(\gamma)$.

AltS then converges if (and only if) $G(\gamma)$ lies in regions 2 or 3 for γ sufficiently close to $\gamma*$ (the fixed point). There are also a number of conditions that are sufficient for AltS to converge from any initial γ (for example, see[1]). Under such conditions, $G(\gamma)$ is necessarily in regions 2 or 3 everywhere.

S. Lui. "On Schwarz Alternating Methods for Nonlinear Elliptic PDEs". In: *SIAM J Sci Comput* 21.4 (1999), pp. 1506–1523.

When does NR accelerated AltS converge in 1D?

Applying NR to $G(\gamma) - \gamma$ gives an accelerated AltS. Like any NR method, we need the slope to take on certain values in certain regions.

$G(\gamma)$ lies in	Necessary condition	Sufficient condition
1	$G'(\gamma) > 1$	
2	$G'(\gamma) < 1$	$G'(\gamma) < 1/2$
3	$G'(\gamma) < 1/2$	$G'(\gamma) < 0$
4	$G'(\gamma) < 0$	

Monotonicity of AltS in 1D

Lemma

As long as the problem to be solved is nonsingular on all subdomains then $G(\gamma)$ is strictly monotonic.

Proof outline: If $G(\gamma_1) = G(\gamma_2)$ then $u_2(x)$ is the same for both γ_1 and γ_2. Therefore, $u_2(\alpha)$ is the same for both $\implies$ $u_1(x)$ is the same for both $\implies$ $u_1(\beta)$ is the same for both.

1D example

Recall the 1D example with Dirichlet boundary conditions on (-1,1):

$$u''(x) - \sin(au(x)) = 0.$$

This equation is nonsingular and admits only the trivial solution $u(x) = 0$.

1D example

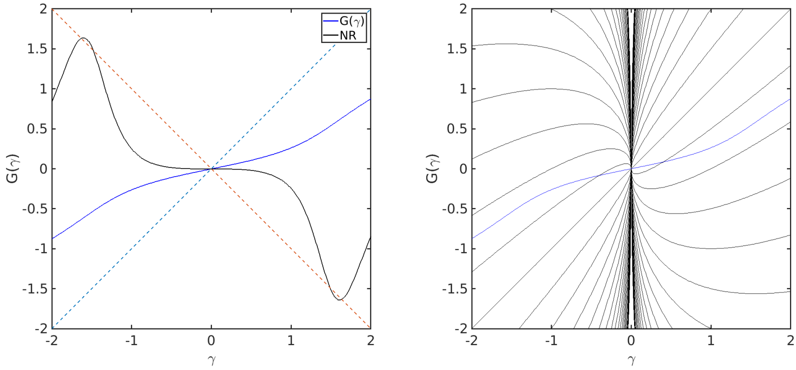

Figure: Left: $G(\gamma)$ and NR accelerated AltS; also plotted are the lines $y = x - x_0$ and $y = 2x_0 - x$. Right: $G(\gamma)$ with the geometry of figure 9.

AltS will converge for this example since $G(\gamma)$ lies entirely within region 2.

However, Newton-Raphson accelerated AltS won't converge for all initial conditions as it crosses the line between regions 3 and 4. There will be a (small) domain where the method converges to a stable oscillation.

1D example

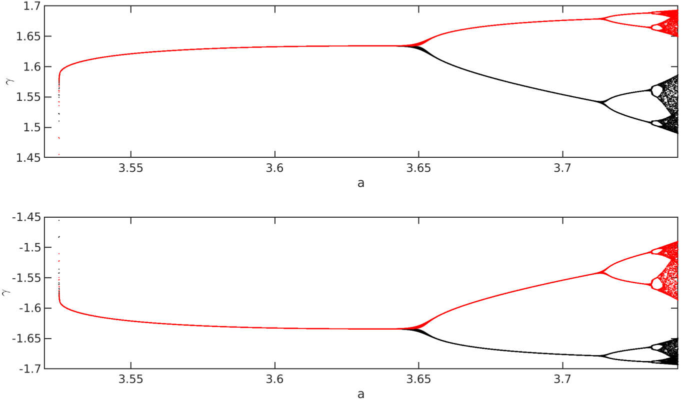

Figure: Period doubling bifurcation in the example caused by NR acceleration.

Changing overlap changes where cycling occurs in parameter space.

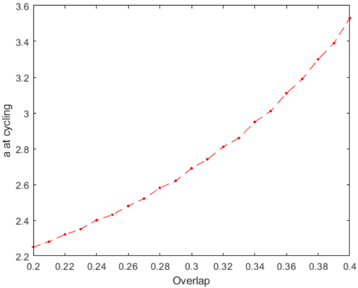

Figure: Parameter at which cycling starts as a function of overlap.

As overlap increases, the basin of cycling grows in parameter space but the number of initial conditions that converge to it dwindles.

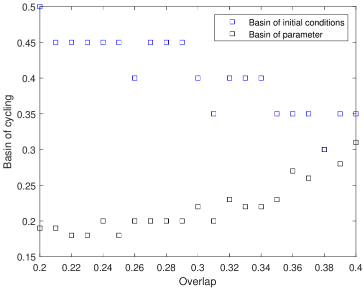

Figure: Basin of cycling as a function of overlap. The basin is in two dimensions: parameter and initial condition.

Possible algorithm

Suppose we know that $G(\gamma)$ lies in FP region 2. Then we can apply an algorithm guaranteed to converge.

1. Select $\gamma_0 \in \mathbb{R}$ and set $n = 0$.
2. Calculate $G(\gamma_n)$ and $G'(\gamma_n)$. If $G'(\gamma_n) = 1$ set $\gamma_{n+1} = G(\gamma_n)$, increment n and repeat this step. If not, proceed to the next step.
3. Calculate the Newton iteration for $G(\gamma_n)$ (using the Davidenko-Branin trick), denoted $\tilde{\gamma}_n$. If $|G'(\gamma_n) - 1| \leq 1/2$ then set $\gamma_{n+1} = \tilde{\gamma}_n$, increment n and return to step 2. If not, calculate the average of γ_n and $\tilde{\gamma}_n$, denoted $\hat{\gamma}_n$, and proceed to the next step.
4. Calculate $G(\hat{\gamma}_n)$. If $G(\hat{\gamma}_n) - \hat{\gamma}_n$ has the same sign as $G(\gamma_n) - \gamma_n$ then set $\gamma_{n+1} = \tilde{\gamma}_n$. If not, set $\gamma_{n+1} = G(\gamma_n)$. In either case, increment n and return to step 2.

2D

The line between regions 3 and 4 in 2D is now defined by all possible rotations around the root.

$$x_{n+1} - x^* = R(x_n - x^*), \quad R^\top R = I$$

If a function $f(x)$ satisfies

$$f(x) = J_f(x)(I - R)(x - x^*)$$

at some point for any rotation matrix R then that point lies on the boundary between regions 3 and 4 and therefore represents a cycle.

2D - damping in y–direction

$$u_{xx}(x, y)+\epsilon u_{yy}(x, y)-\sin(au) = 0$$

If $\epsilon \to 0$ then we retrieve the 1D problem. If ϵ is sufficiently small, we see cycling again.

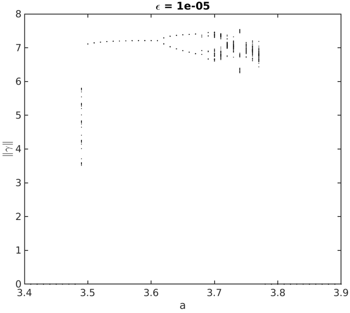

Figure: Bifurcation diagram for $\epsilon = 1e - 5$.

Future Work

- Determine relation between operator parameters (overlap size, tangential diffusion, etc.) and basin of cycling
- Find conditions under which such cycling is impossible
- Improve algorithm for higher dimensions