Nonlinear Preconditioning for the Phase-Field Fracture Model

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Phase-field Fractures

Phase-field energy

Energy in the phase-field model combines elastic potential energy with energy related to the size and shape of any fractures in the material:

$$E(u,v) = \frac{1}{2} \int_\Omega \left((1-v)^2 + k_\ell \right) \sigma_0(u) \cdot \epsilon(u) d\Omega$$
$$+ \frac{G_c}{c_w} \int_\Omega \frac{v^2}{\ell} + \ell |\nabla v|^2 d\Omega,$$

where σ_0 is the stress tensor, ϵ is the strain tensor, ℓ is a regularization parameter, k_ℓ, G_c and c_w are constants, $u \in W^{1,2}(\Omega)$ is the displacement and $v \in W^{1,2}(\Omega; [0,1])$ is the auxiliary variable representing damage on the material.

Minimizing the energy

A solution to the model minimizes this energy:

$$\hat{u}, \hat{v} = \operatorname*{argmin}_{u,v} E(u, v) - f,$$

where f represents external work on the system causing stress, incremented in quasi-static steps to mimic gradually increasing stress. Irreversibility conditions are placed on $\hat{v}$, so that damage isn't 'repaired'.

This minimization problem is non-convex.

Alternate Minimization (AltMin)

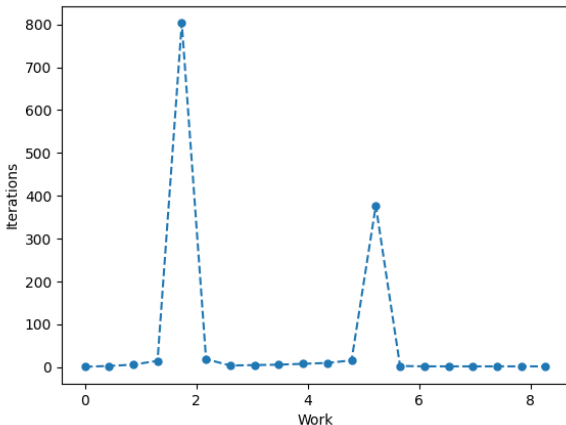**But** these two problems *are* convex:

$$u_{k+1} = \operatorname*{argmin}_{u} E(u, v_k) - f,$$
$$v_{k+1} = \operatorname*{argmin}_{v} E(u_{k+1}, v) - f.$$

This algorithm, AltMin, then has guaranteed convergence to at least a local minimum of $E(u, v)$.

AltMin slow-downs

In practice, AltMin takes a long time at critical moments of the model, usually when cracks begin to propagate. (animation)

Accelerating with Newton's method

Speed-ups in AltMin

The proposed solution to AltMin slow-downs is to accelerate the algorithm with Newton's method. Since AltMin is equivalent to multiplicative Schwarz, the result is MSPIN: Multiplicative Schwarz Preconditioning Inexact Newton.

This approach has shown some effectiveness [1]. However, traditional MSPIN cannot guarantee convergence like AltMin can. This could lead to robustness issues if we're not careful.

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[1] A. Kopanicakova, H. Kothari, R. Krause, **Nonlinear field-split preconditioners for solving monolithic phase-field models of brittle fracture**, Comput. Methods Appl. Mech. Engrg. 403 (2023)

AltMin as fixed point iteration

$$(u_{k+1}, v_{k+1}) = g(u_k, v_k),$$

where $g(x, y)$ represents the two steps of AltMin:

$$g(x, y) = \begin{pmatrix} \underset{u}{\operatorname{argmin}}\, E(u, y) \\ \underset{v}{\operatorname{argmin}}\, E(\underset{u}{\operatorname{argmin}}\, E(u, y), v) \end{pmatrix}.$$

We seek the fixed point of $g(x, y)$:

$$g(\hat{u}, \hat{v}) = (\hat{u}, \hat{v}).$$

AltMin + INK

To accelerate AltMin, consider the function with root at the fixed point:

$$f(x, y) = g(x, y) - (x, y).$$

The Jacobian of this function is

$$J(x, y) = \begin{pmatrix} E_{uu} & \\ E_{vu} & E_{vv} \end{pmatrix}^{-1} \begin{pmatrix} E_{uu} & E_{uv} \\ E_{vu} & E_{vv} \end{pmatrix}$$

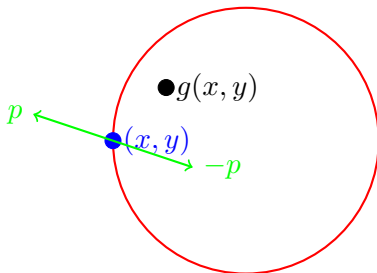We can then approximately solve

$$J(x, y)p = -f(x, y),$$

with an inexact Newton-Krylov method, then setting
$(u_{k+1}, v_{k+1}) = (u_k, v_k) + p.$

We now have two options for iterations from the point (x, y):
$g(x, y)$ (AltMin) and $(x, y) + p$ (MSPIN).

These three points all line within a 2D plane where any subsequent
iteration based on these methods could land. We can then restrict
any analysis to this plane.

Davidenko-Branin trick

The guaranteed convergence of AltMin suggests using the Davidenko-Branin trick [2]:

$$p = \frac{\operatorname{adj} J}{|\det J|} f(x, y)$$

essentially introducing a minus sign if $f(x, y)$ is on the 'wrong' side of the true solution.

We can 'cheat' and determine whether or not to introduce the sign change by seeing if p or $-p$ points more towards $g(x, y)$. In the picture below, we would drop p in favour of $-p$.
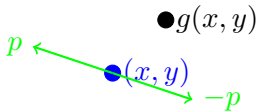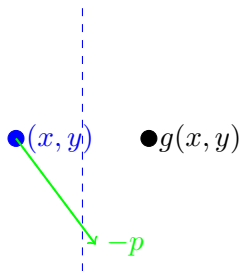

[2]R.P. Brent, **On the Davidenko-Branin method for solving simultaneous nonlinear equations**, IBM J. of R&D, 16(4):434-436 (1972)

Since AltMin is convergent,
$g(x, y)$ lies closer to the solution
than (x, y).

In the example here, the
Newton-Davidenko-Branin step
lies closer to the fixed point step
than to the current iterate,
suggesting we can accept it.

Line search

The fixed point step must have a
reasonable step size, unlike
Newton which may leap a great
distance away. But the Newton
direction may be preferable.

Choose to step in the Newton
direction with step size equal to
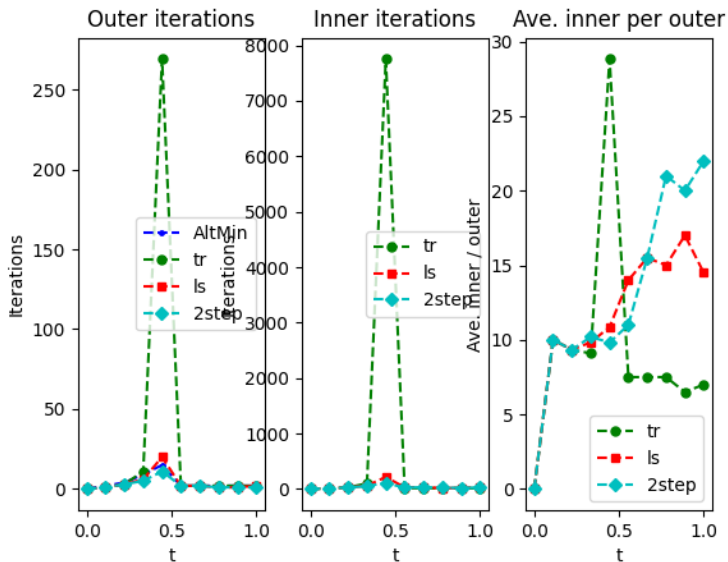the fixed point length, or some
multiple of it.
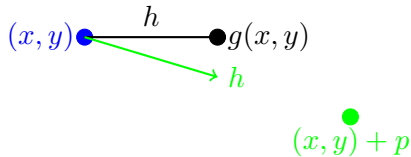
Two-step

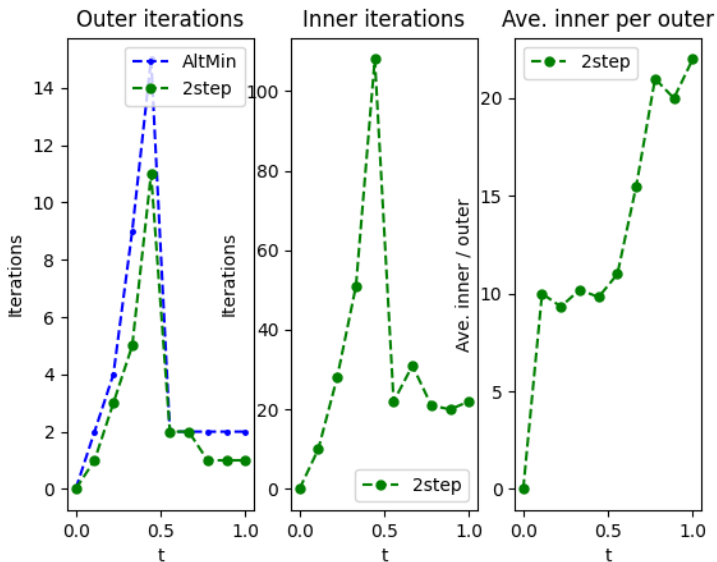Take the fixed point step, then step towards the Newton step, but only part of the way.

We want α to be larger (but at most 1) if the distance between $g(x, y)$ and $(x, y) + p$ is smaller, as we can have more confidence in the results.

Conclusions and future work
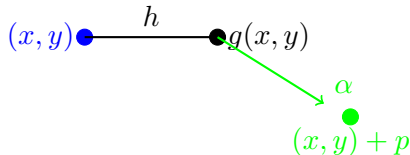
- We can leverage the guaranteed convergence of AltMin to get more robust MSPIN algorithms
- We can develop geometric combinations of fixed point and Newton methods
- More analysis is needed for more sophisticated methods